

Noncommutative Reissner-Nordström Black hole

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A deformed embedding of the Reissner-Nordström spacetime is constructed within the framework of a noncommutative Riemannian geometry. We find noncommutative corrections to the usual Riemannian expressions for the metric and curvature tensors, which, in the case of the metric, are valid to all orders in the deformation parameter. We calculate the area of the event horizon of the corresponding noncommutative R-N black-hole, obtaining corrections up to fourth order in the deformation parameter for the area of the black-hole. Finally we include some comments on the noncommutative version on one of the second order scalar invariants of the Riemann tensor, the so called Kretschmann invariant, a quantity regularly used in order to extend gravity to quantum level.

I. INTRODUCTION

In General Relativity the main assumption is that the space, time and gravity can be modeled as a single entity, the spacetime. General Relativity analyzes spacetime as the background in which, electromagnetism, matter and their mutual influences interact, and it has been used mainly in the study of large scale phenomena.

However it is a general belief that the picture of spacetime as a pseudo Riemannian manifold M locally modeled as the flat Minkowski spacetime, should break down at scales of the order of Planck length, $\lambda_p = (\hbar G/c^3)^{1/2} \approx 1.6 \times 10^{-33}$ cm. A quantum theory of fields, attempting to incorporate gravitation, must consider limitations on the possible accuracy of localization of events in spacetime. A lot of work has been done on the possible mechanisms which could lead to such limitations. The noncommutative geometry has arisen as an option for a description of quantized spacetime. The study of noncommutative geometry acquires relevancy in the research of the quantum nature of spacetime at high energy scales. The idea of noncommutative spacetime coordinates is old and has been in the literature from a long time ago [19]. A successful approach to this topic is provided by the theory of A. Connes, Ref. [9] formulated within the framework of C^* algebras. An important advance in the mathematical framework of noncommutative geometry of recent years was introduced by the deformation quantization of Poisson manifolds by M. Kontsevich Ref. [14]. His work has led on to study new applications of noncommutative geometries to quantum theory. Besides, Seiberg and Witten Ref. [18] showed that the anti-symmetric tensor field arising from massless states of strings can be described by the noncommutativity of the coordinates of spacetime

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu} \quad (1)$$

where $\Theta^{\mu\nu}$ is a constant antisymmetric tensor. Now the multiplication of the algebra of functions is given by the Moyal product

$$(f \star g) = f(x) \exp\left(\frac{i}{2} \Theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x). \quad (2)$$

Many recent investigations are oriented towards a formulation of General Relativity on noncommutative spacetimes. The noncommutative geometrical approach to gravity could give us insight into a theory of gravity compatible with quantum mechanics. Much work has been done in that direction, Refs. [2, 3, 7, 15]. The noncommutative spacetime with the commutation relation given by Eq. (1) violates Lorentz symmetry but it was shown to have quantum symmetry under the twisted Poincaré algebra, see Ref. [3]. The abelian twist element

$$\mathcal{F} = \exp\left(\frac{-i}{2} \Theta^{\mu\nu} \partial_\mu \otimes \partial_\nu\right) \quad (3)$$

was used in Refs.[3, 5] to twist the universal enveloping algebra of the Poincaré algebra providing a noncommutative multiplication for the algebra of functions which is related to the Moyal product. Then it seems natural to extend this procedure to other symmetries of noncommutative field theory.

Related to the noncommutative formulation of General Relativity, Nicolini et al. Ref. [17] found a new solution of the coupled Einstein-Maxwell field equations inspired in the noncommutative geometry. The metric they have found interpolates smoothly between a *de Sitter* geometry at short distances, and a *Reissner-Nordström* (R-N) geometry far away from the origin. Contrary to the ordinary R-N spacetime, in this metric there is no curvature singularity at the origin, neither naked nor shielded by horizons, which seems very intriguing.

In a second paper, see Ref. [1], the same authors solved Einstein-Maxwell equations in the presence of a static spherically symmetric Gaussian distribution of mass and charge having a minimal width. They show that the coordinate fluctuations can be described, within the coherent states approach, as a smearing effect.

In this letter we are going to follow the theory of noncommutative Riemannian geometry developed by Chaichian et al. in Ref. [4] to investigate quantum aspects of gravity from a mathematical point of view. In Ref. [4] a noncommutative Riemannian geometry is constructed by developing the geometry of noncommutative n -dimensional surfaces. The notions of metric and connections are introduced on such noncommutative sur-

faces, giving rise to the corresponding Riemann curvature. Chaichian et al. makes use of Nash's theorem of isometric embeddings —Refs. [10, 16]—in order to obtain the deformed versions of classical spacetimes. In the same framework of noncommutative geometry, Wang et al. [22] have constructed a quantum Schwarzschild spacetime and a quantum Schwarzschild-de Sitter spacetime with cosmological constant. They computed the metrics and curvatures and finally showed that, up to second order in the deformation parameter, the quantum spacetimes are solutions of the so called noncommutative Einstein equations. In the present work we will construct a noncommutative deformation (which can be interpreted as quantum corrections) of the Reissner-Nordström spacetime and we will look for properties of such noncommutative spacetime. The key results are the equations of the metric and the components of the Riemann tensor. A preliminary version of this work (exploring a different approach) was presented in the IX Workshop of the Gravitation and Mathematical Physics Division of the Mexican Physical Society, [20].

The paper is organized as follows. In Section II we give a quick review of the deformation of the algebra of functions on the domain of Euclidean space. In section III we explain how we “deform” the Riemannian geometry by calculating the relevant quantities in General Relativity. This particular section is based upon Chaichian et al. [4]. In Section IV the Reissner-Nordström spacetime is introduced. We find the components of the noncommutative versions of the metric and Riemann tensors. We compute the area of the event horizon and find its noncommutative corrections. Besides, we search for the generalization of the Kretschmann invariant (which provides us information about the embedding) and we find its noncommutative expression. Finally in Section V we give our conclusions and perspectives.

II. DEFORMING THE ALGEBRA OF FUNCTIONS

In this section we will introduce the deformation of the geometry. Keeping this in mind let's begin by introducing a noncommutative algebra denoted by \mathcal{A}_{\hbar} , called *deformed algebra*, satisfying a *correspondence principle* which establishes that we recover the commutative algebra for $\lim_{\hbar \rightarrow 0} \mathcal{A}_{\hbar} = \mathcal{A}$.

There are different ways to deform a theory. Here we follow the approach given by Wang et al. in Ref. [22] and we use the Riemannian structures previously studied in Ref. [4], using the Moyal product of functions. The next subsection is based on the pioneering work of Gerstenhaber, Ref. [11], who studied the deformation of algebras.

A. Deformation of the Algebra

Let \mathcal{A} be a commutative ring with the unit. The Ring of formal power series $R = \mathcal{A}[[X]]$ is the set of all the sequences (a_0, a_1, a_2, \dots) with $a_i \in \mathcal{A}$ for all $i \in \mathbb{N}$ where the operations of sum and product are defined as:

$$1. \quad (a_0, a_1, a_2, \dots) + (b_0, b_1, b_2, \dots) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots).$$

$$2. \quad (c_0, c_1, c_2, \dots) = (a_0, a_1, a_2, \dots) \times (b_0, b_1, b_2, \dots),$$

where $c_j = a_0 b_j + a_1 b_{j-1} + \dots + a_j b_0$ for all $j \in \mathbb{N}$.

It is verified immediately that R is a commutative ring with a unit $1 = (1, 0, 0, \dots)$. We write $X = (0, 1, 0, 0, \dots)$, such that $X^2 = (0, 0, 1, 0, \dots)$, etcetera.

We'll say that $\alpha = (a_0, a_1, a_2, \dots)$ is of order i when $a_0 = a_1 = \dots = a_{j-1} = 0$ and $a_i \neq 0$. We write $\mathcal{O}(\alpha) = i$.

We have given a precise definition of a formal power series. From now on we are going to use the notation $\sum_{i \geq 0} a_i X^i$ instead of (a_0, a_1, a_2, \dots) . It is clear that this is not a sum. Besides, $\mathcal{A}[[X]]$ is a subring of R .

Let's denote by U a certain domain in \mathbb{R}^n and let \mathcal{A} be the set of all the formal power series in \hbar with coefficients on the real functions C^∞ on U . Then \mathcal{A} is a $\mathbb{R}[[\hbar]]$ -module and its elements are formed as $\sum_{i \geq 0} f_i \hbar^i$, where the explicit product of two elements from the $\mathbb{R}[[\hbar]]$ -module is:

$$\left(\sum_{i \geq 0} f_i \hbar^i \right) \left(\sum_{i \geq 0} g_i \hbar^i \right) = \sum_{n \in \mathbb{N}} \left(\sum_{k \geq 0}^n f_k g_{n-k} \hbar^n \right)$$

Up to this point we have given the algebra that we are going to use to quantize (which means: to deform) the spacetime. Now we will proceed to deform the product of functions substituting the usual product by a noncommutative one, the so called *Moyal product*. Let $f, g : U \rightarrow \mathbb{R}$ be two functions C^∞ , we denote by fg the usual (i.e. commutative) product of functions. Then we define the star product (or Moyal product) as the operation $\star : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that:

$$(f \star g) = f(x) \exp \left(\hbar \sum_{i,j} \Theta_{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) g(x) \quad (4)$$

where the exponential function $\exp(\hbar \sum_{i,j} \Theta_{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j)$ must be understood as a power series in the differential operator and Θ_{ij} is a constant antisymmetric tensor represented by a matrix of $(n \times n)$, where n is the dimension of U . It is easy to see that the star product is associative and satisfies distributivity too. (\mathcal{A}, \star) is the so called deformed algebra.

Obviously we can construct more complex structures with the deformed algebra: let's denote by $\mathcal{A}^m = \mathcal{A} \oplus \mathcal{A} \oplus \dots \oplus \mathcal{A}$, (m times), with elements (X_1, X_2, \dots, X_m) , where $X_i \in \mathcal{A}$, with the index i running from 1 to m .

III. DEFORMATION OF THE GEOMETRY

Now we will construct an embedding X of a pseudo Riemannian manifold of dimension n to a pseudo Euclidean space of dimension m and with a metric tensor

$$\eta_{\alpha\beta} = \text{diag}(\underbrace{1, 1, \dots, 1}_p, \underbrace{-1, -1, \dots, -1}_q), \quad (5)$$

with $p + q = m$, such that $X = (X^1, X^2, \dots, X^m) \in \mathcal{A}^m$. This section is based entirely in Ref.[22].

We can construct tangent vectors denoted by E_i

$$E_i \equiv \partial_i X, (i = 1, 2, \dots, n) \quad (6)$$

at each point of the surface given by the parametrization X . Now we will define the metric tensor as an $(n \times n)$ matrix:

$$\mathbf{g}_{ij} = E_i \bullet E_j, \quad (7)$$

where the fat dot \bullet denotes *inner product* between elements of the algebra

$$\bullet : \mathcal{A}^m \otimes \mathcal{A}^m \rightarrow \mathcal{A}^m \quad (8)$$

on the open region U by

$$A \bullet B = \sum_{i=1}^p a_i \star b_i - \sum_{j=p+1}^{p+q} a_j \star b_j, \quad (9)$$

for every $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_m)$ in \mathcal{A}^m . Obviously \mathbf{g}_{ij} is invertible in U if and only if $\mathbf{g}_{ij}|_{\hbar=0}$ is invertible, and we denote the inverse matrix \mathbf{g}^{ij} , as the matrix which satisfies

$$\mathbf{g}_{ij} \star \mathbf{g}^{jk} = \mathbf{g}^{kj} \star \mathbf{g}_{ji} = \delta_i^k,$$

with δ_i^k the identity $n \times n$ matrix.

In this noncommutative spacetime the relevant connection is given by

$$\nabla_i E_j = \Gamma_{ij}^k \star E_k,$$

where the corresponding expression for Γ_{ij}^k is

$$\Gamma_{ij}^k = \partial_i E_j \bullet \tilde{E}^k. \quad (10)$$

and $\tilde{E}^k = E_i \star \mathbf{g}^{ik}$. The respective noncommutative Riemann tensor is given by:

$$\mathbf{R}_{kij}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^p \star \Gamma_{ip}^l - \Gamma_{ik}^p \star \Gamma_{jp}^l, \quad (11)$$

In terms of the deformed product, two different contractions of the Riemann tensor can be defined: the *Ricci curvature* tensor \mathbf{R}_j^i and the *Θ_j^i curvature tensor*:

$$\mathbf{R}_j^i = \mathbf{g}^{ik} \star \mathbf{R}_{kpj}^p, \quad (12)$$

$$\Theta_p^l = \mathbf{g}^{ik} \star \mathbf{R}_{kpi}^l. \quad (13)$$

In general \mathbf{R}_j^i and Θ_p^l do not coincide, in contrast to the commutative case.

Now the noncommutative *curvature scalar* of the surface X is:

$$\mathbf{R} = \mathbf{R}_i^i \quad (14)$$

On the other hand the noncommutative Einstein equations with cosmological constant on U are:

$$\mathbf{R}_j^i + \Theta_j^i - \delta_j^i \mathbf{R} + 2\delta_j^i \Lambda = 2\mathbf{T}_j^i, \quad (15)$$

where T_j^i is some generalized *energy-momentum* tensor, Λ is the cosmological constant. This equation reduces to the Einstein's equation in vacuum when $T_j^i = \Lambda = 0$.

IV. THE REISSNER-NORDSTRØM SOLUTION

In the next subsection we give a brief review of the so called Reissner-Nordstrøm solution. First we give a summary of the structures on the standard commutative case. In order to compare it with the noncommutative case, we show the components of the metric tensor, the Christoffel symbols, Riemann tensor and the scalar curvature.

Then in the following subsection we will find the corresponding noncommutative Riemannian structures introduced in section 3 for the case of R-N spacetime.

A. Commutative Reissner-Nordstrøm Spacetime

The Reissner-Nordstrøm solution represents the spacetime outside a static, spherically symmetric charged body carrying an electric charge. It is the unique spherically symmetric asymptotically flat solution of the Einstein-Maxwell equations.

The metric of the R-N spacetime can be written as:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (16)$$

where $m = \frac{2GM}{c^2}$ represents the gravitational mass, with G the Newton's constant and c the speed of light. Here e stands for the electric charge.

If $e^2 > m^2$ the metric is non-singular everywhere except for the irremovable singularity at $r = 0$. If $e^2 \leq m^2$ the metric has singularities at r_+ and r_- where $r_{\pm} = m \pm (m^2 - e^2)^{\frac{1}{2}}$. For a especially succinct review see Ref. [12]. In this letter we will be interested only in the region outside r_+ . Here, we will denote the coordinates by: $\mathbf{x}_1 = r$, $\mathbf{x}_2 = \theta$, $\mathbf{x}_3 = \varphi$, $\mathbf{x}_4 = t$, (notice the particular choice for the coordinate t). Now, the respective Christoffel symbols $\Gamma_{ij}^k = g^{kl} \Gamma_{ijl}$ for this spacetime are

given by the well known expressions:

$$\begin{aligned}
\Gamma_{14}^4 &= \Gamma_{41}^4 = -\Gamma_{11}^1 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right), \\
\Gamma_{22}^1 &= -r \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), \\
\Gamma_{33}^1 &= -r \sin^2 \theta \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), \\
\Gamma_{44}^1 &= \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right), \\
\Gamma_{12}^2 &= \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = r^{-1}, \\
\Gamma_{33}^2 &= -\sin \theta \cos \theta, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \cos \theta / \sin \theta
\end{aligned} \tag{17}$$

And the nonzero components of the Riemann tensor are:

$$\begin{aligned}
R_{212}^1 &= R_{242}^4 = -r \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right), \\
R_{313}^1 &= R_{343}^4 = -r \sin^2 \theta \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right), \\
R_{414}^1 &= \left(\frac{-2m}{r^3} + \frac{3e^2}{r^4}\right) \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), \\
R_{323}^2 &= \sin^2 \theta - \sin^2 \theta \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), \\
R_{121}^2 &= -r^{-1} \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right) \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}, \\
R_{424}^2 &= R_{434}^3 = r^{-1} \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right) \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), \\
R_{131}^3 &= r^{-1} \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right) \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}, \\
R_{232}^3 &= 1 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), \\
R_{441}^4 &= -\left(\frac{-2m}{r^3} + \frac{3e^2}{r^4}\right) \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1},
\end{aligned} \tag{18}$$

whereas the nonzero components of the Ricci tensor are:
 $R_{11} = -\frac{e^2}{r^4} \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}$, $R_{22} = \frac{e^2}{r^2}$, $R_{33} = \frac{e^2}{r^2} \sin^2 \theta$, $R_{44} = \frac{e^2}{r^4} \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)$. And the scalar curvature is $R = 0$.

B. Noncommutative Reissner-Nordström Spacetime

We are going to compute the deformed Riemannian structures given in section III for the R-N spacetime. By following the procedure depicted in the preceding section, we propose that the Reissner-Nordström spacetime can be embedded in a six dimensional pseudo Euclidean space with metric $\eta_{\mu\nu} = \text{diag}(1, 1, 1, 1, -1, -1)$, through the

map $X = (X_1, X_2, X_3, X_4, X_5, X_6)$ given by

$$\begin{aligned}
X_1 &= g(r) \\
X_2 &= r \sin \theta \cos \varphi \\
X_3 &= r \sin \theta \sin \varphi \\
X_4 &= r \cos \theta \\
X_5 &= \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}} \sin t \\
X_6 &= \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}} \cos t
\end{aligned} \tag{19}$$

with the convention of coordinates given in the previous subsection. Here we have used the Kasner embedding already used before by Wang and Zhang in Ref.[22] for the case of a noncommutative embedding of a Schwarzschild and a Schwarzschild-de Sitter spacetime in a 6 dimensional pseudo Euclidean manifold. Using a similar procedure than that implemented by Wang and Zhang, we define the function $g(r)$ in the first of the Eqs.(19) as a smooth function such that

$$g'^2 + 1 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} \left(1 + \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right)^2\right)$$

where g' denotes the derivative of g with respect to r . This requirement for $g(r)$ simplifies the computations of the components of the noncommutative metric as will be clear very soon. From the Eqs. (16) and (19) it is easy to verify the isometry of the embedding:

$$\begin{aligned}
ds^2 &= (dX_1)^2 + (dX_2)^2 + (dX_3)^2 + (dX_4)^2 \\
&\quad - (dX_5)^2 - (dX_6)^2
\end{aligned} \tag{20}$$

We now proceed to deform the algebra of functions with the procedure mentioned in section III and imposing a Moyal product of functions with the matrix representation of the antisymmetric tensor $\Theta_{\mu\nu}$

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

The tangent vectors $E_i = \partial_i X = \frac{\partial}{\partial x_i} X$ are given by:

$$\begin{aligned}
E_1 &= \left[g'(r), \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta, \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-\frac{1}{2}} \right. \\
&\quad \left. \times \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right) \sin t, \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-\frac{1}{2}} \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right) \cos t \right] \\
E_2 &= [0, r \cos \theta \cos \varphi, r \cos \theta \sin \varphi, -r \sin \theta, 0, 0] \\
E_3 &= [0, -r \sin \theta \sin \varphi, r \sin \theta \cos \varphi, 0, 0, 0] \\
E_4 &= \left[0, 0, 0, 0, \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}} \cos t, -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}} \sin t \right]
\end{aligned} \tag{22}$$

And from Eq. (7) we find Ref.[20] the nonzero components of the metric tensor :

$$\begin{aligned}
\mathbf{g}_{11} &= \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} - \cos 2\theta \sinh^2 \hbar \\
\mathbf{g}_{12} &= \mathbf{g}_{21} = r \sin 2\theta \sinh^2 \hbar \\
\mathbf{g}_{13} &= -\mathbf{g}_{31} = -r \sin 2\theta \sinh \hbar \cosh \hbar \\
\mathbf{g}_{22} &= r^2 + r^2 \cos 2\theta \sinh^2 \hbar \\
\mathbf{g}_{23} &= -\mathbf{g}_{32} = -r^2 \cos 2\theta \sinh \hbar \cosh \hbar \\
\mathbf{g}_{33} &= r^2 \sin^2 \theta - r^2 \cos 2\theta \sinh^2 \hbar \\
\mathbf{g}_{44} &= -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)
\end{aligned} \tag{23}$$

In agreement with the observation of Chaichian et al. Ref.[6], in the sense that, for an arbitrary $\Theta_{\mu\nu}$ given by Eq.(21), the deformed metric $\mathbf{g}_{\mu\nu}$ is not diagonal. In the components of $\mathbf{g}_{\mu\nu}$ we notice that terms containing the

noncommutative parameter appear. Actually the hyperbolic functions—depending on the noncommutative parameter \hbar —enter when we compute the deformed inner product, Eq. (7), in the form of series expansions. It is a straightforward calculation to verify that the components of the noncommutative metric tensor satisfy the correspondence principle. It is important to remark that the components $\mathbf{g}_{12}, \dots, \mathbf{g}_{33}$ coincide with those computed by Wang et al [21]. That coincidence in some of the components of the metric tensor is due to the map defined by Eq. (19). In order to make these calculations more clear, we now proceed to compute in detail one of the components of the metric, for example \mathbf{g}_{11} . From Eq.(7) we know that

$$\mathbf{g}_{11} = E_1 \bullet E_1,$$

and using the expressions for the generators obtained before, then after some simplifications of terms, we get

$$\begin{aligned}
\mathbf{g}_{11} &= g'^2(r) + \sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + 2 \frac{\hbar^2}{2!} [(\sin^2 \theta \cos^2 \varphi - \cos^2 \theta \sin^2 \varphi) + (\sin^2 \theta \sin^2 \varphi - \cos^2 \theta \cos^2 \varphi)] + \\
&+ 8 \frac{\hbar^4}{4!} [(\sin^2 \theta \cos^2 \varphi - \cos^2 \theta \sin^2 \varphi) + (\sin^2 \theta \sin^2 \varphi - \cos^2 \theta \cos^2 \varphi)] + \dots + \cos^2 \theta - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right)^2
\end{aligned} \tag{24}$$

Now using the fact that

$$g'^2 + 1 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} \left(1 + \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right)^2\right)$$

and identifying the series $\sinh^2 \hbar = 2 \frac{\hbar^2}{2!} + 8 \frac{\hbar^4}{4!} + 32 \frac{\hbar^6}{6!} + \dots$; we finally obtain the component $\mathbf{g}_{11} = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} - \cos 2\theta \sinh^2 \hbar$. Now from Eq. (7) and using the result $\mathbf{g}^{kj} \star \mathbf{g}_{ji} = \delta_i^k$, we can calculate the contravariant components of the noncommutative metric tensor now. Those components turn a little bit more complicated than those in Eq.(23), precisely because their expressions involve the determinant of the noncommutative metric. Fortunately the components of the covariant metric tensor $\mathbf{g}_{\mu\nu}$ just involve functions of the coordinates r and θ and then, the Moyal products between them turn into the usual commutative products. In the following computations it will be very convenient to introduce the function $P(r)$ which is the metric coefficient of the timelike coordinate in the line element (Eqn.16) defined by $P(r) \equiv \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)$. Now we can see that all the components of the noncommutative contravariant metric tensor will be affected by the presence of the charge in the R-N black hole, because of its dependence on the determinant of the noncommutative metric and on the function $P(r)$ defined before.

Now, from Eq. (10) we can calculate the respective components of the noncommutative connection $\Gamma_{ij}^k = \partial_i E_j \bullet \tilde{E}^k$. Taking into account the next definitions:

$P'(r) = \frac{dP}{dr} = \frac{2m}{r^2} - \frac{2e^2}{r^3}$, and $P''(r) = -\frac{4m}{r^3} + \frac{6e^2}{r^4}$, the modified connections will be written in the form $\Gamma_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha + \hbar f_1(x^\mu) + \hbar^2 f_2(x^\mu) + \dots$, up to second order in the deformation parameter by :

$$\begin{aligned}
\Gamma_{11}^1 &= \Gamma_{11}^1 + \frac{P'}{2} (P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) \hbar^2, \\
\Gamma_{22}^1 &= \Gamma_{22}^1 + [rP(P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) + 2 + 4 \cos 2\theta] \hbar^2, \\
\Gamma_{33}^1 &= \Gamma_{33}^1 + rP[P \sin^2 \theta (P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) + 2(\cos^4 \theta + 1) + 4 \cos 2\theta] \hbar^2, \\
\Gamma_{44}^1 &= \Gamma_{44}^1 + \frac{PP'}{2} (P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) \hbar + \mathcal{O}(\hbar^2) \\
\Gamma_{12}^2 &= \Gamma_{12}^2 + \frac{1}{r} (\csc^2 \theta \cos 2\theta + 2 \cot \theta \cos 2\theta) \hbar + \mathcal{O}(\hbar^2), \\
\Gamma_{33}^2 &= \Gamma_{33}^2 + [2P \cos^3 \theta \sin \theta - P \cos \theta (-P^{-1} \cos 2\theta \cos^2 \theta + 2 \sin^2 \theta) + 4 \cot \theta \cos 2\theta + 2 \sin 2\theta] \hbar^2, \\
\Gamma_{13}^3 &= \Gamma_{13}^3 - \frac{P}{r} (\cos 2\theta + 1) \hbar + [-3P(1 + \cos^2 \theta) + \cos 2\theta \times (3 - \cot^2 \theta + 2 \cot \theta) - 2 \cot \theta (\sin 2\theta + 2 \cos \theta)] \hbar^2, \\
\Gamma_{23}^3 &= \Gamma_{23}^3 + P \cot \theta (\cos 2\theta + 1) \hbar + [\cot \theta (-3P - 2 \cot \theta \sin 2\theta + 3 \cos 2\theta - 2P \cos^2 \theta - 7 \cot^2 \theta)
\end{aligned}$$

$$+ 2 \frac{\cos(2\theta)}{\sin \theta} + \frac{2}{\sin^2 \theta} - 2 \Big) + 4P \cos \theta \sin \theta \Big] \hbar^2$$

and finally,

$$\Gamma_{14}^4 = \Gamma_{14}^4. \quad (25)$$

where we have used the fact that, by definition, Γ_{ij}^k is symmetric in the covariant indices. This last statement is clear from Eq. (10). It is interesting to compare the expressions that we have just obtained in Eq. (25) with those, previously listed in Eq. (17). We can easily verify that the correspondence principle is satisfied. With the last expressions given in Eq. (25) we calculate the components of the noncommutative Riemann tensor.

In order to compare with the respective commutative expressions—Eq.(18) in subsection IV A—we focus only those components. However there are more components of the Riemann tensor which are purely noncommutative and don't have a commutative counterpart. For the sake of clarity in the expressions we will write one of the components of noncommutative Riemann tensor in the form $\mathbf{R}_{\beta\mu\nu}^\alpha = R_{\beta\mu\nu}^\alpha + \hbar f_1(x^\mu) + \hbar^2 f_2(x^\mu) + \dots$

$$\begin{aligned} \mathbf{R}_{rtr}^t &= -\partial_r \Gamma_{rt}^t + \Gamma_{rr}^r \star \Gamma_{rt}^t - \Gamma_{rt}^t \star \Gamma_{rt}^t \\ &= R_{rtr}^t + \frac{P^{-1}P'^2}{4}(1 + 2\cos^2 \theta)\hbar^2 \end{aligned} \quad (26)$$

As in the case of the noncommutative connections, we can verify that the correspondence principle is satisfied for the limit $\hbar \rightarrow 0$. For the rest of the noncommutative Riemann tensors, the calculations are straightforward but the expressions rapidly become cumbersome in some cases. For many of the noncommutative Riemann tensors, the first nonvanishing corrections appear at second order in \hbar . However, some of the purely noncommutative Riemann tensors have expressions which depend on first order terms of \hbar .

One useful quantity that we can calculate is the Kretschmann scalar $K = \mathbf{R}_{\alpha\beta\mu\nu} \star \mathbf{R}_{\alpha\beta\mu\nu}$. This invariant has been used in the pursue to extend gravity to quantum level.

In our case there are some of the noncommutative components of the totally covariant Riemann tensor that contribute to first order in the perturbative parameter \hbar . For example, those depending on the connections Γ_{44}^1 , Γ_{12}^2 , Γ_{13}^3 and Γ_{23}^3 written in Eq.(25). Thus we expect corrections of order \hbar in $\mathbf{R}_{1\beta\mu\nu}$, $\mathbf{R}_{2\beta\mu\nu}$ and $\mathbf{R}_{3\beta\mu\nu}$. So first order corrections to the Kretschmann scalar come from cross products of the usual contravariant Riemann components with their respective covariant first order corrections and vice versa.

After performing this long calculation we find that

$$K_{NC} = K + f(r, \theta, \phi)\hbar$$

where $f(r, \theta, \phi)$ comprises the first order corrections of all the noncommutative Riemann tensors and K is the usual commutative expression reported in ([8, 13])

At this point we can obtain various expressions for different physical quantities in terms of the deformation parameter. Although the Hawking temperature of the R-N black hole is not modified by the embedding, we can ask us about the area of the event horizon in the deformed R-N Black Hole. This is given by the integral

$$A = \int_{r=r_+} \sqrt{\det \mathbf{g}_{ab}} d\theta d\phi$$

with $a, b = 2, 3$, and r_+ stands for the exterior radius of the event horizon $r_+ = m + \sqrt{m^2 - e^2}$. From Eq.(23), we write

$$\mathbf{g}_{ab} = \begin{pmatrix} \mathbf{g}_{22} & \mathbf{g}_{23} \\ \mathbf{g}_{32} & \mathbf{g}_{33} \end{pmatrix} \quad (27)$$

Which leads to the following result

$$A = \int_{r=r_+} r^2 \sin \theta \sqrt{1 - \cos 2\theta \sinh^2 \hbar} d\theta d\phi, \quad (28)$$

Performing the integral in θ we obtain the following expression in terms of the parameter \hbar

$$\begin{aligned} &\sqrt{(1 - \sinh^2 \hbar)} + \frac{\sqrt{2}}{2 \sinh \hbar} \arctan \left(\frac{\sqrt{2} \sinh \hbar}{\sqrt{1 - \sinh^2 \hbar}} \right) \\ &+ \frac{\sqrt{2}}{2} \sinh \hbar \arctan \left(\frac{\sqrt{2} \sinh \hbar}{\sqrt{1 - \sinh^2 \hbar}} \right) \end{aligned}$$

Which is a smooth function of \hbar . In order to compare with the usual commutative result, this can be expanded in powers of the deformation parameter using the series expansion of $\sinh^2 \hbar$ given before and the corresponding expression for the function $\arctan(x)$. Then, for the area of the event horizon of the deformed R-N black hole we obtain

$$A = 4\pi r_+^2 \left(1 + \frac{\hbar^2}{6} - \frac{\hbar^4}{360} + \mathcal{O}(\hbar^6) \right) \quad (29)$$

This result is similar to that obtained by Wang and Zhang in Ref.[22]. After all this is not surprising, because the determinant of \mathbf{g}_{ab} has the same expression as that reported in [22]. That is due to the particular embedding we have used which inherits the symmetry of Schwarzschild solution over the angular coordinates. It is easy to verify that we obtain the regular—commutative—expression for the case in which $\hbar \rightarrow 0$. However, in the present case, the radius of the event horizon is obviously modified by the charge of the black hole. Then, for the expressions of the components of the Riemann tensor—see for example Eq.(26)—we have nontrivial modifications because of the presence of the functions $P(r)$ and its derivatives, which contain explicitly the charge. In the same way we expect nontrivial modifications for the deformed Einstein field equations Eq.(15).

V. CONCLUSIONS AND PERSPECTIVES

As we have mentioned before, there are different ways to construct a noncommutative geometrical approach to gravity. In this letter we have followed the prescription given by Wang and Zhang in Ref.[22]. We don't find corrections to the surface gravity of the noncommutative RN black hole: $\kappa = 1/2\partial_1 g_{44}|_{r=r_+}$, so the Hawking temperature

$$T_H = \frac{\sqrt{m^2 - e^2}}{2\pi(m + \sqrt{m^2 - e^2})^2}$$

is not modified by the embedding, in contrast to Ref. [1] where a correction to T_H is given in terms of the noncommutative parameter for a variety of charged objects. On the other hand, we have found an analytic expression for the noncommutative corrections to the event horizon area, related to the entropy of the R-N black hole. We developed the calculations of the noncommutative Ricci tensors, but the mathematical expressions that we obtain

become very heavy quite soon.

For the case of the deformed versions of Einstein field equations, Eq.(15), it is not clear how to construct a suitable noncommutative stress tensor T_j^i . A possible solution to this problem would be to proceed like in Ref. [22], working with approximate expressions—given in terms of powers of the noncommutative parameter, in the same way that we have proceeded in this letter—for the components of the Ricci tensor. This will be studied in a future paper.

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